Mechanics of masonry vaults: The equilibrium approach

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ABSTRACT: The theory of masonry structures should take into account the essentials of the material "masonry": heterogeneity, good compressive strength, almost no resistance to tension, and a high friction coefficient. Besides, it should be appropriated to the usual structural type of old masonry buildings, i.e., vaulted structures with massive buttresses. Finally, it should consider that cracks are present in most masonry buildings and that these cracks may vary with time. From the end of the seventeenth century a "scientific" theory of vaulted masonry structures has been developed. Professor Heyman has incorporated this "old" theory of masonry structures within the broader frame of modern Limit Analysis. This scientific theory was preceded by another: the traditional "geometrical" theory of the old master builders. Both theories tried to solve the fundamental problem of structural design: to design safe structures, i.e., to understand what makes an structure safe (or unsafe). Both theories arrive to same conclusion: the safety of a masonry structure 0is a matter of geometry. A safe state of equilibrium is achieved through a correct geometry. Both historically and theoretically the "equilibrium approach" is the best approach to the analysis and design of masonry structures.

1. INTRODUCTION

Father Vicente Tosca, architect, philosopher, mathematician, and astronomer, . . . begins his Treatise of Architecture (a part of his *Compendio mathematico*, 9 vols. 1707-1715) treating the subject of vault design :

> The most subtle and exquisite part of Architecture . . . is the formation of every sort of Arches and vaults, cutting their stones, and adjusting them which such artifice, that the same gravity and weight which should have precipitate them to the earth, maintain them constantly in the air, supporting one another in virtue of the mutual complication which links them, and in such a way close above masonry buildings with all safety and strength.

Equilibrium is achieved through geometry and, in this way, is possible to construct safe masonry buildings. Indeed, the old traditional rules for the design of masonry vaults and buttresses are geometrical, in the sense that they define certain proportions between the structural elements (for example, the thickness of the buttress is a certain fraction of the vault's span). In fact, in Toscas affirmation we find the essence of the structural design of masonry.

But for us, architects and engineers of the beginning of the XXIth century, all this appears too naive. Maybe a proof of the ignorance of the old masterbuilders; indeed, until the XVIIIth century a true scientific structural science, based on the findings of the Strength of Materials and the Laws of Mechanics, have not existed. However, these "ignorant" masterbuilbers built The Pantheon of Rome, Hagia Sophia and the gothic cathedrals.

So it may be that the traditional geometrical approach is not so bad, after all. Maybe the old masters have a theory, of a different kind, but based on a deep insight in the nature and behaviour of masonry structures. If this is so, we would like to learn something of this theory, which, if we are to judge by the results, was extraordinary. However, if it is possible to add to our knowledge, it is impossible to subtract from it. We are forced to approach the subject of masonry vault and arch analysis with a "scientific" approach, within the frame of contemporary structural theory. Before proceed further, two observations should be made.

The first concerns the objective of the Theory of Structures. The aim of structural theory is to be able to design safe buildings or to estimate the safety of existing ones. It is an "applied science" not a "pure science". As Rankine (1858) remarked, if the question of the scientist is "what I want to know" the question of the engineer or architect is "what I want to do". Theoretical considerations are conditioned by the fact that we need a response with no delay.

The second refers to our own ignorance of the subject. Masonry vaults are not built any more. The whole tradition of constructing in masonry is lost in the western world. This most "subtle and exquisite" part of architecture is alien to us. Many architects and engineers have not ever seen the building of even the simplest masonry vault. We lack the "feeling" of the builder who selects the stone, design the templates to cut it, plan the centering, directs the order of construction and, finally, command the decentering. Rodrigo Gil de Hontañón (Treatise of architecture, ca. 1540, copied in García, 1681; English translation in Sanabria, 1984) after describing the construction of a gothic cross vault warns that:

> . . .these things may be difficult to understand if one lacks experience and practice, or if one is not a stone mason, or has never been present at the closing of a rib vault.

This is precisely the actual situation of any architect or engineer.

In what follows, we will try to show that traditional and modern views of vault analysis arrive to the same fundamental conclusion: the overall importance of geometry. It will be shown that the modern theory Limit Analysis of Masonry structures, which has been developed mainly by Professor Heyman, is the best tool to understand and analyse masonry buildings. This theory leads to the "approach of equilibrium"; the analyst needs only to study possible equilibrium states with the masonry in compression. The existence of these possible states of equilibrium depends on the geometry. A "safe" building is an "equilibrated" or "balanced" building: modern theory conduces to the same geometrical statements of the traditional theory. It could have not been otherwise, the spectacular successes of the old architects could not have been a matter of chance.

Numerous historic references are made. The intention is to make clear that there is an old tradition of scientific calculation of masonry structures using the approach of equilibrium. There is much to learn from the architects and engineers of the past. They may not have had a perfect grasp of the theory but they do have the essential knowledge, which supply the practice. "Ars sine scientia nihil est", practice is nothing without theory, but theory without practice is simply dangerous. Our "practice" is in the existing buildings and in what we may infer from the careful reading of the Old Treatises of Architecture and Engineering.

As it will be evident by the references this paper is not original. The intention of the author is, simply, to insist in what is for him essential for the design, analysis and intervention in the field of masonry structures. The main frame of the theory and many of the arguments here exposed are due to professor Heyman, with whom the author has an immense intellectual debt.

2. THE MASONRY STRUCTURE: THE MATERIAL

Our aim is to be able to understand and analyse masonry vaults, in fact, any combination of them, i.e., a masonry building. Fig. 1 reproduces a drawing by Viollet-le-Duc of a medieval building, accompanied by two details of wall construction (medieval and roman). The drawings are so detailed that the internal structure can be seen easily. Behind regular ashlar masonry we find a most irregular internal structure. Homogeneity, isotropy, uniform mechanical properties, etc., all the common assumptions of modern conventional structural analysis cannot be made in this case without violence to the most basic common sense.

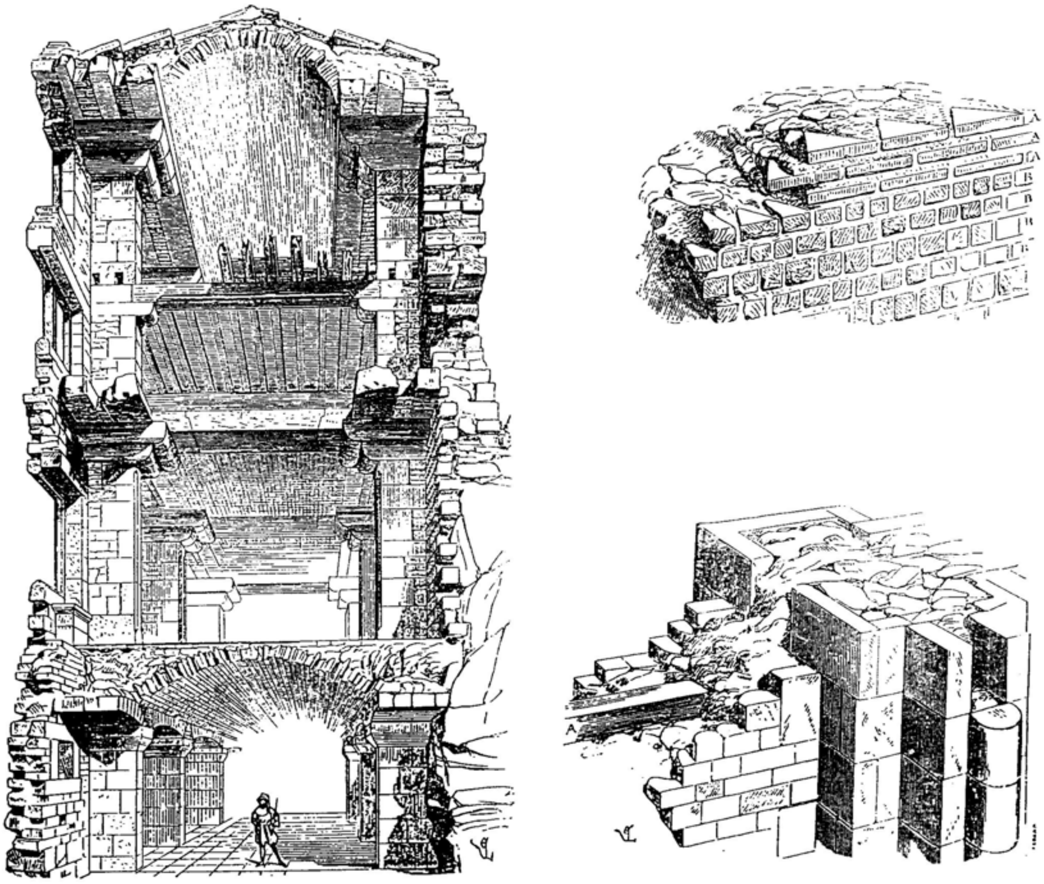

Figure 1 : Constructive section through a medieval building. Details of the construction of roman and medieval walls (Viollet-le-Duc 1858)

What we have is a pile of stones, bricks or rubble, received with mortar or with dry joints, disposed in such a way that they are in equilibrium under the force of gravity. The mortar, when exists, is very weak in tension, so the interaction between the different elements must be through compressive forces. Besides, it is a fact that the building maintains its form during the years: the force of friction between the different elements is sufficiently high to avoid sliding (the typical angle of friction for stone is 30°-35°). Finally, the compressive stresses are usually very low, so that the danger of a failure by crushing is negligible. Maybe this last remark will appear unlikely to a modern architect or engineer, considering the enormous size, which some masonry structures have. But, in fact, in this huge buildings the stresses are an order of magnitude below the crushing values. The mean stress at the base of the columns which support the dome of St. Peters in Rome (dome plus drum weight 400.000 kN) is only 1.7 N/mm. Benouville found that in the pillars of the central nave in Beauvais the mean stress was only of 1.3 N/mm, etc.

We have, then, a composite heterogeneous material, with a great strength in compression, very weak in tension and with no danger of sliding between the stones. All this affirmations were explicitly or implicitly made in the vault theory of the XVIIIth and XIXth centuries, and has been systematised by professor Heyman as the Principles of Limit Analysis of masonry structures. Their importance will be apparent later.

3. THE MASONRY ARCH

3.1 Equilibrium: lines of thrust

The fundamental structural element in masonry architecture is the arch: it is the "natural" way to span a void with a no-tension material. Arches do not exist in nature: it is an invention, which appeared in Babylon perhaps 6.000 years ago (Aztecs and Incas built in masonry for centuries without knowing the arch).

In Fig. 2 we can see a massive Etruscan voussoir arch. Heavy stones have been cut in form of wedges; then they have been placed on a formwork (a centering) beginning from the extremes. When the last stone in the middle (the key-stone) was placed the centering was removed and the arch stood.

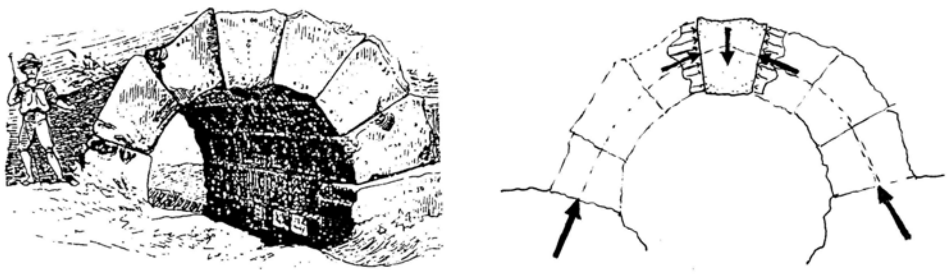

Figure 2 : Etruscan voussoir arch (Durm, 1885)

Let us consider the free-body equilibrium of the keystone. En each joint (which we imagine more or less plane) there will exist a certain stress distribution. The stress resultant must be a compressive force, a "thrust"; the point of application is the "centre of thrust" and it must be contained within the plane of joint. The two thrust in the joints maintain the keystone in equilibrium.

The same occurs with the other stones until we arrive at the springings of the arch. There the abutment must supply/resist a certain thrust. This is the "thrust of the arch" and the abutment must have adequate dimensions to resist it. The masonry arch always push outwards, "the arch never sleeps," says an old proverb attributed to the Arabs. Masonry architecture has, then, two main problems: to design arches that will stand and buttresses which resist their thrust. In fact, the most critical problem is the second because it involves the collapse of the whole structure (most of the traditional design rules were about buttress design).

The locus of the centre of thrust forms a line, the "line of thrust." The form of this line depends, therefore, on the geometry of the arch, its loads and, also, on the family of plane joints considered (the concept was first rigorously formulated by Moseley (1835); an excellent mathematical treatment in Milankowitch (1907).

Of course, to respect the main property of the masonry material, the line of thrust must be contained within the masonry arch. We may imagine one voussoir acting against the other two voussoirs only through the centres of thrust. If we now invert the arch, what was a force of compression will be a tension force: the voussoirs are hanging like a chain, as it appears in Fig. 3 of Robison (1851).

This was Hooke's brilliant idea ca. 1670, when he was trying to solve the problem of the figure and thrust of the arches: "As hangs the flexible line, so but inverted will stand the rigid arch". If the arch has all the voussoirs of the same size the line of thrust have very nearly the form of an inverted catenary. Some twenty years later Gregory (1697) completed Hooke's affirmation: "none but the catenaria is the figure of a true or legitimate arch or fornix. And when an arch of any other figure is supported, it is because in its thickness some catenaria is included" (Heyman 1999). Therefore in Fig. 4 the arch is in equilibrium with an internal stress distribution represented by the inverted catenary.

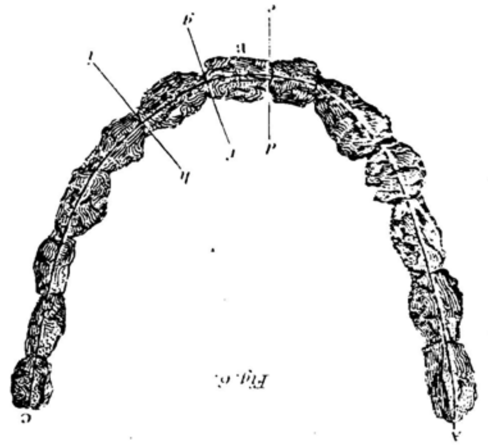

Figure 3 : "Hanging" masonry arch (Robison, 1851)

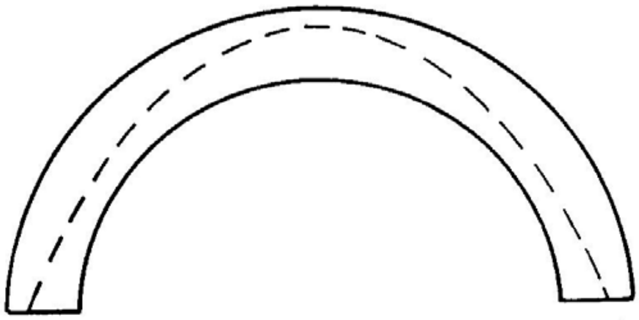

Figure 4 : Possible line of thrust of a semicircular arch (Heyman, 1995)

3.2 Analysis: What is the actual line of thrust?

Any line of thrust within the arch is a possible equilibrium solution. But this solution is not unique. It is evident that, in an arch of sufficient thickness, there are infinite possible inverted catenaries or lines of thrust. The arch is a statically indeterminate (hyperstatic) structure. The equations of equilibrium are not sufficient to obtain the inner forces.

What is, then, the actual line of thrust? Though arch theory was well developed by the end of the XVIIIth century (Heyman 1972), the question was posed for the first time by Moseley. And he tried to determine the position of the line of thrust. To do this he need to make more affirmations, besides these of equilibrium. In the "pre-history" of elastic analysis he enunciated a "new Theorem in Statics" with the purpose of obtaining the reactions of rigid hyperstatic structures: the Principle of Least Pressure (formulation in Moseley 1833); application to arches in Moseley (1843). Applying this principle to arches he concluded that the actual thrust must be the minimum. Moseley's approach reached wide diffusion in Europe. Similar intents were made by Culmann and others (Kurrer 1990). Another approach was to design the arch with the profile of the line of thrust and, then, it was supposed it would coincide with the middle-line of the arch, Villarceau (1853). Other possibility was the physical insertion of hinges, which make the arch statically determinate, i.e., determine the position of the line of thrust (three hinges are needed and in the second half of the XIXth century many masonry bridges were designed tri-articulated). Other times, the hinges were "imagined" by the analyst in order to obtain a certain position of the line of thrust. But, all these approaches were felt to be incomplete, an escape from a defective theory.

3.3 Elastic analysis

Poncelet (1852) was conscious of the problem and in his historical review of arch theory suggested to apply the elastic theory to masonry arches in order to obtain a unique solution (the theory for circular arches made of wood or iron has been developed by Bresse, 1848). However the engineers showed a certain resistance to assimilate masonry (as we have seen, essentially heterogeneous, anisotropic, irregular,. . .) with an elastic material (uniform, isotropic, etc.). In fact, until ca. 1880, engineers divided arches into "elastic", made of wood or wrought iron, and "rigid", made of masonry. Already in the 1860=s some elastic analysis of masonry arches were made (for example by the Spanish engineer and architect Saavedra, 1860). Castigliano (1879) applied his theory of elastic systems also to masonry bridges.

But it was Winkler (1879) who made the first discussion in depth of the elastic approach to masonry arch analysis. After a revision of all the contemporary theories, he concluded that elastic analysis was the best option. However, he added a discussion on the "Störungen" (perturbations) that can affect the position of the line of thrust. Their main origins were: the deformation of the centering during construction, the yield of the buttresses under the thrust and the effect of changes of temperature. All these perturbations will produce some cracking of the arch and Winkler was well aware this will affect notably the position of the line of thrust, which could be very different from the calculated (elastically); he, then, suggested some means of controlling the position of the line of thrust by inserting internal hinges during construction, Figure 5.

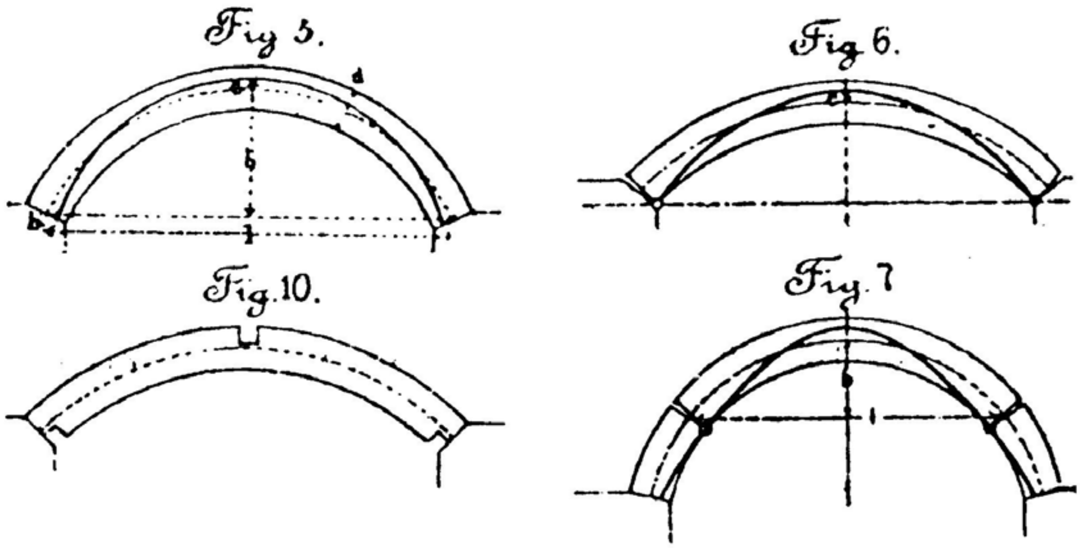

Figure 5 : Displacement of the line of thrust due to some yielding of the buttresses. Devices to "fix" the line of thrust during construction (Winkler, 1880)

After 1880 engineers accepted elastic theory, and the efforts were directed to simplify the heavy calculations involved (Hertwig 1941). However, some doubts still existed and to investigate the application of the theory to masonry or concrete arches the Österreichisches Ingenieur- und architekten-Verein (Austrian Institution of Engineers and Architects) made a complete series of tests on arches of stone, brick, unreinforced concrete and reinforced concrete (some of them of 23 m of span). The results were interpreted as the definitive experimental confirmation of the "modern" elastic theory (see, for example, Howe, 1906). However, in the photographs and drawings of this comprehensive report (more than 130 pages) could be clearly seen the cracking due to movements and mechanisms of collapse (Fig. 6) by the formation of what were called later plastic hinges.

But the engineers of the end of the XIXth were looking for the actual, true solution and elastic analysis appeared to be best option. Therefore, although the masonry arches cracked visibly during construction and/or after the decentering, although the material was irregular, anisotropic and discontinuous, elastic analysis was considered the best theory. It was named "the modern theory of arches" in contrast with the "old theory".

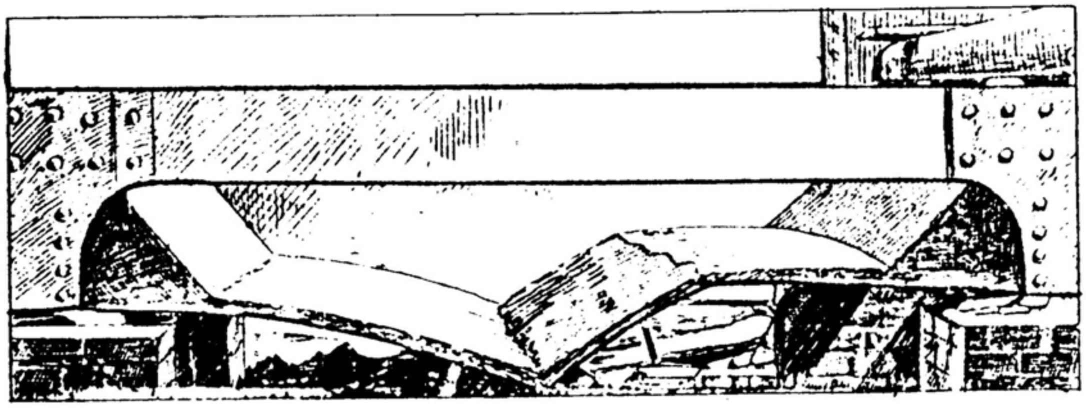

Figure 6 : Test to destruction of a concrete vault (Öst. Ing.- und Arch.-Verein 1895)

3.4 Response to little movements of the abutments. Impossibility of knowing the actual line of thrust

Elastic analysis seems, indeed, to be very rational. There are three main steps involved (Heyman 1999): first, the equilibrium equations are written; secondly, elastic equations are written, relating the internal forces with the deformations of the structure (for example, the bending moment is proportional to the curvature); finally, some statements about the compatibility of deformation are made (affirmations about the way the elements are connected and about the boundary conditions: for example, that the abutments of the arch are encastré). The resultant system of equations can be solved and a unique, elastic, solution obtained. Then stresses are calculated and compared with some admissible values, obtained dividing the crushing strength of the material obtained in laboratory tests.

What is not commonly realised is that the resultant system of equations is very sensible to small changes in the boundary conditions. Professor Heyman has discussed this problem in depth in many publications, which are listed in the references, and we will resume his main arguments.

Let us consider a masonry arch over a centering, Fig. 7. After the decentering the arch begin to thrust against the abutments. Real abutments are not rigid and they will yield a certain amount. The span, then, increases and the arch must accommodate itself to this increment of the span. In what way could an arch (made of a the rigid-unilateral material described above) do this? The arch cracks. A crack opens at the keystone (downwards) and two other cracks open at the abutments (upwards).

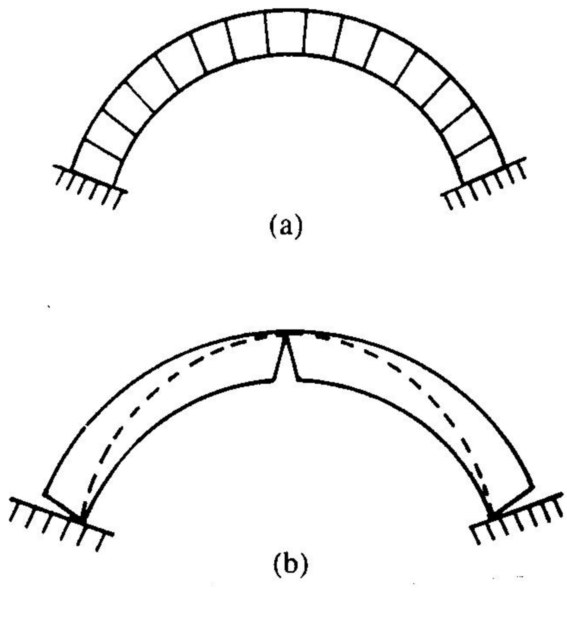

Figure 7 : Cracking of a masonry arch after decentering (Heyman 1995)

The arch becomes tri-articulated and a unique line of thrust is possible. But it may be, that the movement is not symmetrical: perhaps the right abutment besides yielding horizontally, yields vertically. To every possible movement corresponds a certain cracking, and cracks open and close to permit the arch to respond to this aggression of the environment. This may be observed using models. Even simple, "plane", cardboard models give very good results, Fig. 8.

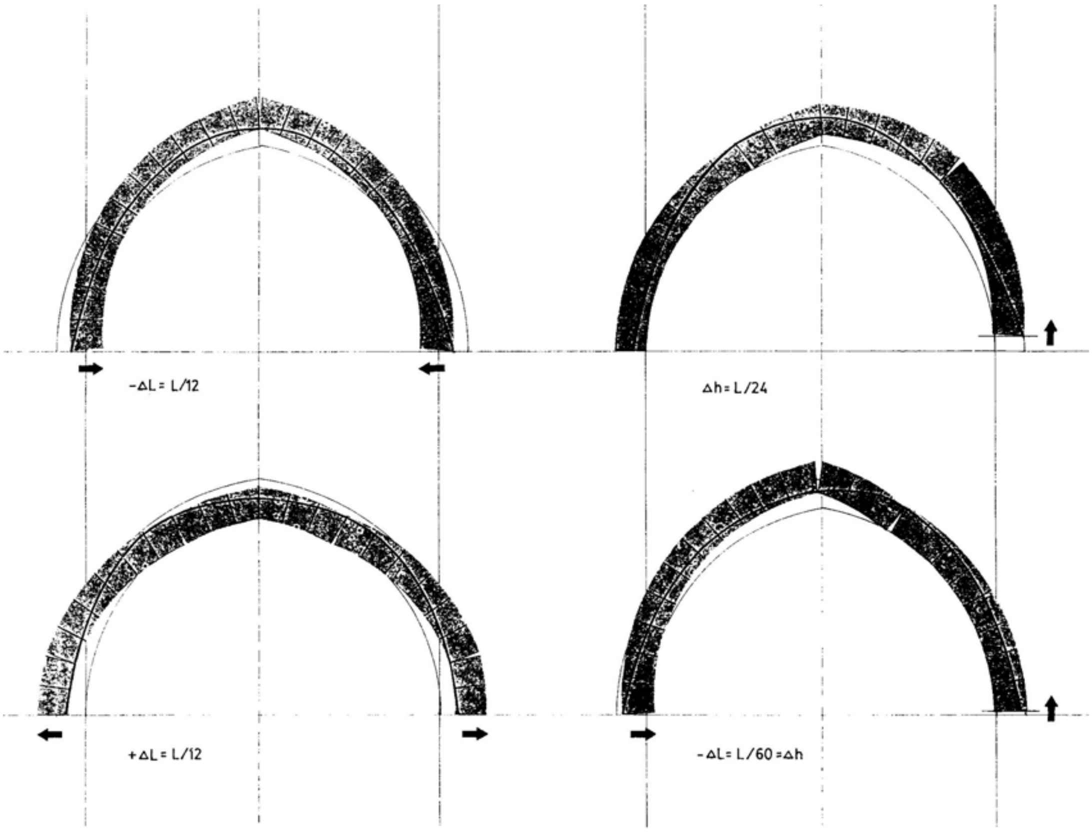

Figure 8 : Different patterns of cracking due to movements of the abutments (observed in a cardboard model Huerta 1990, Huerta 1996)

Cracks are, then, not dangerous. The capacity of the structure to respond to the aggressions of the environment resides precisely in the possibility of cracking and, this depends on the statements made about the material: infinite compressive strength, no resistance to tension and impossibility of sliding.

The cracking determines the position of the line of thrust. As the cracking varies the line moves abruptly from one position to another (i.e., the internal forces change completely), Fig. 8. In the model the movements are very large but even small movements, impossible to appreciate by inspection, have the same effect. As it is obviously impossible to know or predict these kind of perturbations, in fact, it is essentially impossible to know what is the actual line of thrust, i.e., in what state is the arch. We know, however, that whatever the line of thrust, it must be contained within the arch.

Though it is impossible to know the actual thrust of the arch, it is possible to establish its value within certain limits. There are two extreme positions of the line of thrust, which corresponds to the minimum thrust and to the maximum thrust, as it is evident in the Fig. 9.

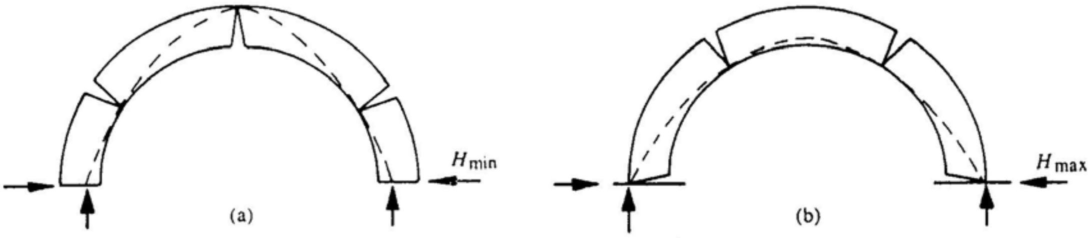

Figure 9 : Semicircular arch under its own weight. a) Minimum thrust; b) maximum thrust (Heyman 1995)

The cracks function like hinges and it is precisely the material properties cited in paragraph 2, which allow hinge formation. This concept of "hinge" is crucial to the understanding of masonry structures.

In particular, deformations are not "elastic" in any sense; they are the result of the division of the structure in ascertain number of parts which, connected through the hinges, allow certain movements. In Fig.10, the original semicircular barrel vault is severely distorted due to an increase of 250 mm of the original span of 6.5 m. The movements were stopped with the addition of massive buttresses, and it is clear the danger of collapse by "snap-through". Cracks are not dangerous, but great unrestricted displacements of the abutments can, of course, lead to the catastrophic collapse of the structure.

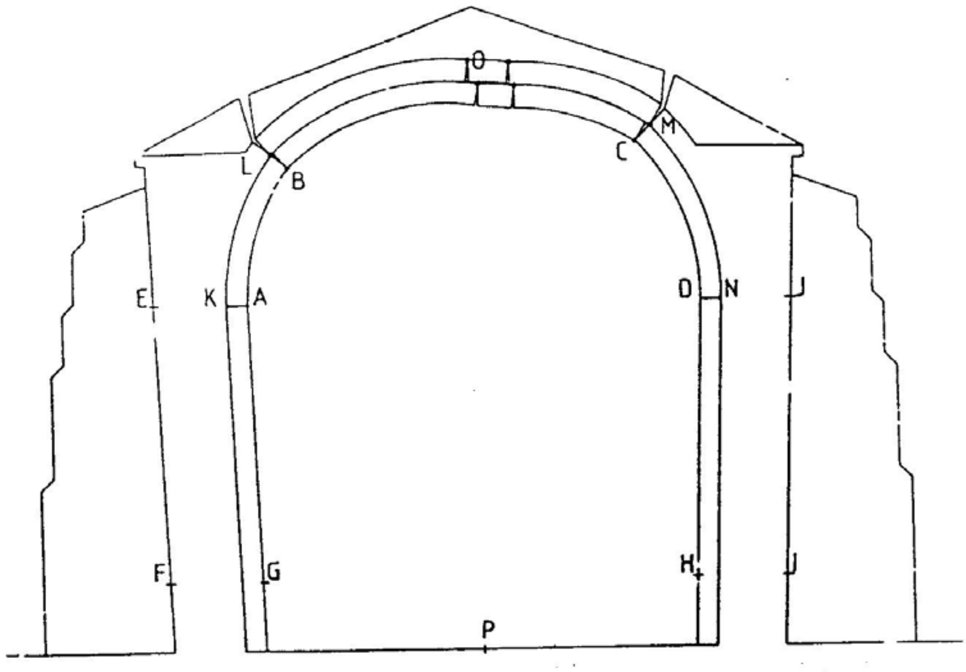

Figure 10 : Barrel vault with gross deformations. The deformed state cannot be explained "elastically" (Huerta/López 1997)

3.5 Collapse of arches

To understand completely masonry arch behaviour the collapse of arches should be studied. And the naive question arises, as how is it possible that a structure built with an infinitely strong material can collapse. We have seen that excessive deformation can lead to collapse. But, will it be possible the collapse with no movement of the abutments?

As we have seen when the line of thrust touches the limit of the masonry a "hinge" forms, which allows the rotation. Three hinges make the arch statically determined and, as we have seen, an arch with three hinges is a stable structure. One more hinge, however, will convert the arch in a four-bar mechanism which will collapse. Therefore an increase of the load which will lead to the formation of four hinges will lead to collapse without crushing of the material. This can occur in a stable arch with addition of load, which deforms sufficiently the line of thrust. Again the hanging chain analogy makes the process clear, Figure 11.

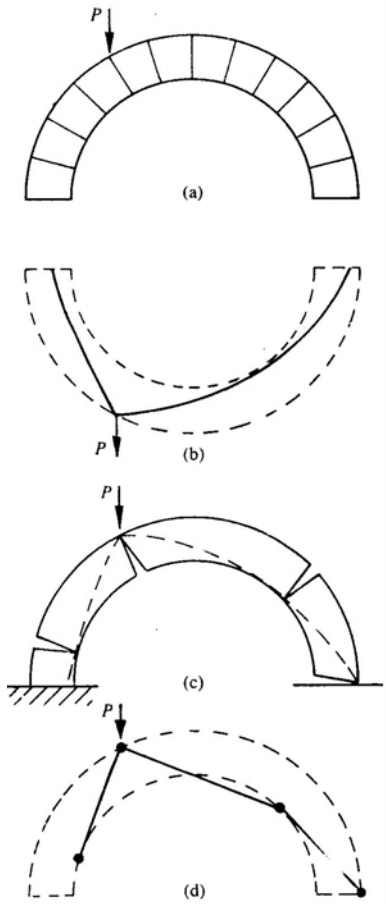

Figure 11 : Collapse of a semicircular masonry arch under a point load (Heyman 1995)

3.6 Limit analysis of masonry arches. The fundamental theorems

If we can draw a line of thrust within the arch we know that this arch will have at least one possibility to stand. But, does this mean that the arch will stand? Will it not be possible to find, also, a possible way for the arch to collapse? Exists the possibility that a little unforeseen movement, could cause a cracking, which will lead to collapse?

This is the central question of structural design. The engineers and architects of the XIXth century worked with the assumption that it was enough to design the structure with a certain degree of safety, with reference to a certain equilibrium state. For example, Rankine (1858) said that an arch would be safe if it is possible to draw a line of thrust within its middle-third. But he gave no formal proof. Then, the design will achieve the maximum of safety when the central line of the arch is made to coincide with the line of thrust. Of course, catenarian arches also crack after decentering due to unexpected little movement of the abutments. However, the fact is that the method worked: bridges and buildings designed in such way stood firmly for decades or centuries.

The solution of the problem came only in the XXth with the theory of Limit Analysis and the demonstration of the Fundamental Theorems (Gvozdev 1936, 1960). There is no space here to explain the origins and development of Limits Analysis, and the reader is directed to the books and articles of professor Heyman (see especially Heyman 1998, 1999). In particular the Safe Theorem states that if it is possible to find an internal system of forces in equilibrium with the loads which does not violate certain material assumptions, the structures will not collapsed, it is "safe". In the case of the masonry arch, any line of thrust compatible with the applied loads will satisfy the equilibrium conditions. The material requirements are the Principles cited in paragraph 2, the main requirement being the absence of any tensile forces. Therefore, if it is possible to draw a line of thrust (equilibrium) within the arch (no-tension material) this is an absolute proof that the arch is stable and that collapse will never occur.

No affirmations are made about boundary conditions. The arch will crack as in Fig. 8 responding to movements of the abutments, the line of thrust will move markedly finding new equilibrium states, but it will never go out of the masonry and it will never form a sufficient number of hinges to convert the arch in a mechanism. (Of course if the structure is severely distorted, as in Fig. 10, the calculations must be made with reference to the actual distorted geometry.)

Therefore, the Safe Theorem of Limit Analysis solves the problem of finding the actual line of thrust. It is impossible to know the actual line of thrust, but this is unimportant, as we can calculate the safety of the structure without making assumptions about its actual state.

3.7 The safety of masonry arches

The fundamental theorems permit also to calculate the safety of masonry arches. Professor Heyman as proposed a geometrical factor of safety obtained comparing the geometry of the actual arch with that of the "limit arch" which will just support the loads. With reference to Fig. 12 of Heyman, it is evident that by the Safe Theorem the arch in (a) will be safe; a possible line of thrust is comfortably within the masonry. Now, if we diminish the thickness of the arch for certain value it will be possible to draw only one line of thrust contained within the arch. The line touches (due to the symmetry) in five points, we have then five hinges and the arch is in unstable equilibrium and will collapse. We can establish the safety of the original arch comparing its thickness which that of the limit arch. If the actual arch has double thickness the geometrical factor will be 2, and so on. In the case of a bridge, the limit arch for the worst position of the load should be found (Heyman 1982).

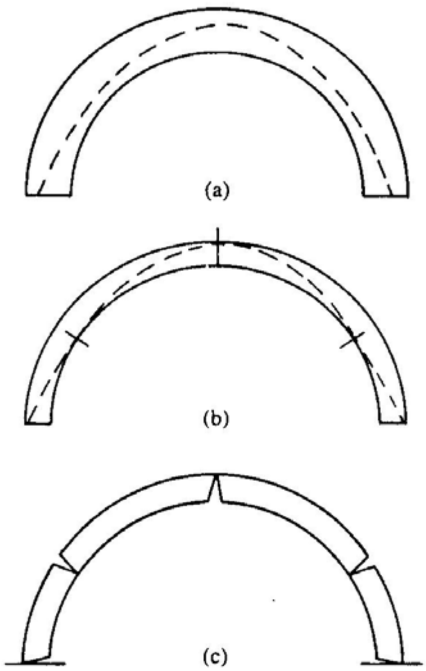

Figure 12 : Semicircular arch: a) stable; b) of limit minimum thickness

To obtain the exact value of the geometrical factor of safety can involve heavy calculations. But to obtain a lower bound may be very easy. For example, to show that for a certain arch under certain loads the geometrical factor is equal or greater than 2, it will suffice to draw a line of thrust within the middle-half of the arch. In the second half of the XIXth century arches were designed so that it will be possible to draw a line of thrust within the middle-third. The elastic justification of avoiding any tension in the joints has, as we have seen, no sense, but the procedure was completely safe.

4. MASONRY VAULTS: THE EQUILIBRIUM APPROACH

One of the most important results of the Safe Theorem is that it permits an "equilibrium approach" to the analysis of structures (Baker and Heyman 1969; Heyman 1995 and in many references cited). The task of the analyst is not to find the actual equilibrium state, but to find reasonable states of equilibrium for the structure under study. In fact, this has been the approach of all the great architects and engineers. It was implicit in the "geometrical design" of the masterbuilders. It was explicit in the work of Maillart, Torroja, Nervi, Candela or Gaudi, to cite only a few great engineers and architects.

The approach of equilibrium permits the analysis of complex vaults with reference to the arch theory already exposed. The technique consists in imagine the vault divided in a series of arches and to look for a line of thrust inside each one of these arches. If this is possible, we have found a possible equilibrium solution in compression and, by the Safe Theorem, the structure is safe.

4.1 Domes

A dome can be imagined as composed by a series of arches obtained slicing the dome by meridian planes. Every two "orange slices" form an arch; if it is possible to draw a line of thrust within this arch, then we have found a possible equilibrium state in compression and the dome is safe, it will not collapse. This "slicing technique" applied to domes was first, implicitly, employed by Bouguer (1734) and Frézier (1737), explicitly, applied it to many types of vaults (including domes) in a qualitative way. The first calculation of an actual dome imagined composed of arch-slices were made in the context of the expertises about the dome of Saint Peters, Rome, by the "three mathematicians" applying for the first time the principle of virtual work in structural analysis (Jacquier, Le Seur, Boscovitch 1743) and Poleni (1743, 1748) who was the first to use models. After this, the division in arches cutting with meridian planes became the usual approach to dome analysis. Professor Heyman (1967, 1977) has revived the technique and, for the first time, has explained the theoretical assumptions and applications within the frame of Limit Analysis.

Collapse analysis of domes is much more complicated. The first studies of the limit thickness of domes were first made by Kobell (1855) and collapse mechanisms for domes by Beckett (1877), but they are not entirely correct. The first rigorous study has been made by Heyman (1967, 1977); see also Oppenheim et al. (1989). The theme is, in general, of no interest in the analysis of historic structures, because we are more concerned with equilibrium and safety.

As for the cracks, the typical pattern to be found in most cases is that of meridian cracks. This is produced by a slight yielding of the buttress system (of the tambour wall in most cases). Again, it has been professor Heyman the first to make a systematic rigorous study of the problem (Heyman, 1988). The dome thrust outwards and the masonry of the abutment system gives way radially. Meridian cracks form inevitably as it is seen in Fig. 13. A non-symmetrical movement can produce patterns similar to that shown in Fig.14. It is possible to calculate the height of the cracks and, of course, to relate the movements of the different parts. Heyman (1988) have done this for the dome of the roman Pantheon. Dome analysis is, then, due to symmetry, a quite simple affair. If the dome has a high tambour, its stability should be checked. In Fig.15 the stability of a masonry dome which follows the geometrical rules of Fontana (1694) has been checked (supposing the dome of one shell and tambour of dome of the same material).

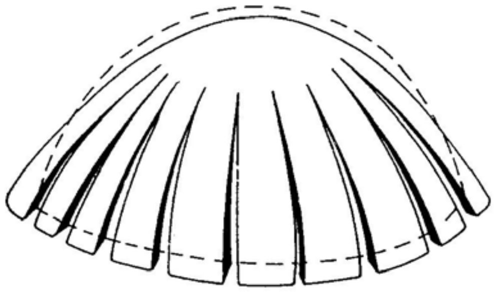

Figure : 13 Typical cracking of a dome
(Heyman 1988)

Figure14 : Partial cracking of a dome
(Heyman 1988)

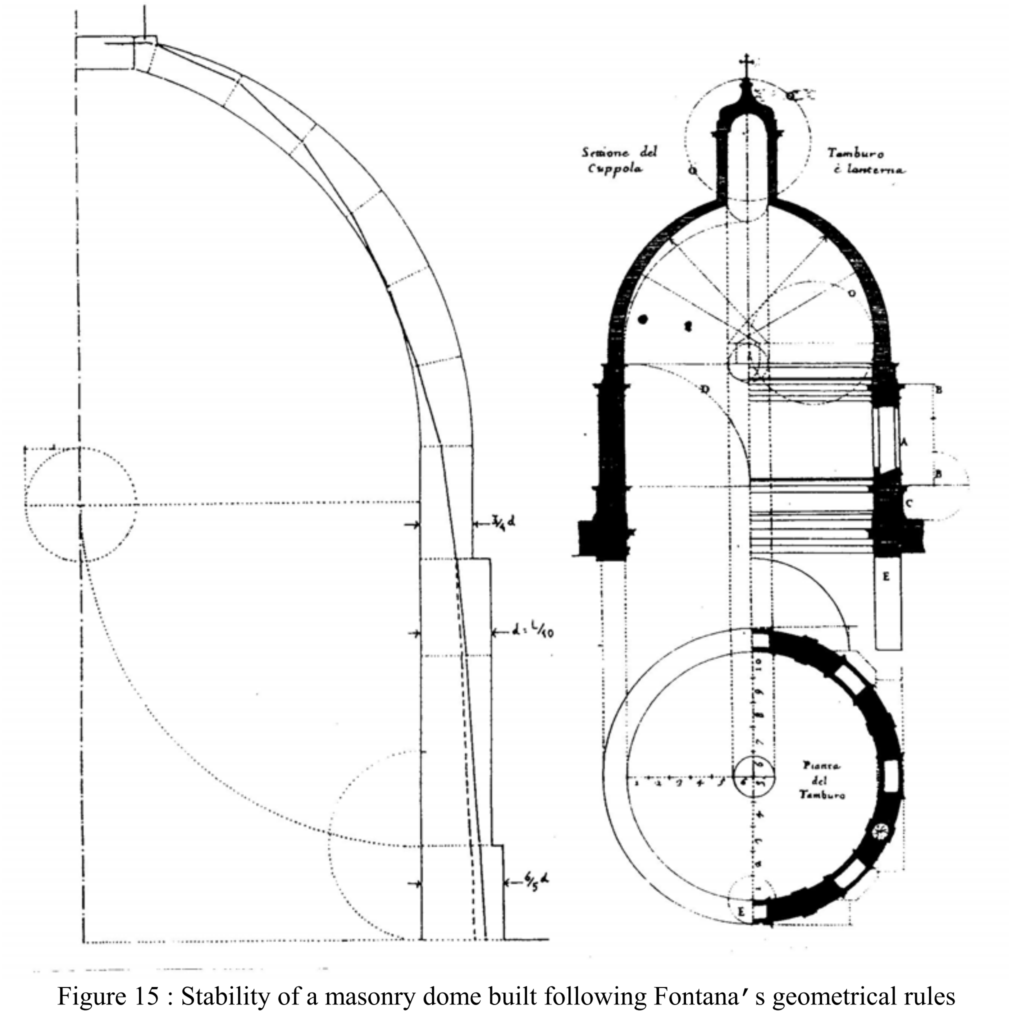

Figure 15 : Stability of a masonry dome built following Fontana's geometrical rules

4.2 Gothic vaults

As with domes the slicing technique can be applied to gothic vaults. In this case the pattern of slicing depends on the form of the vault. Let us consider first a typical cross vault resulting of the intersection of two pointed barrel vaults. Now we can imagine each barrel vault as made of a series of elemental arches which rest upon the cross ribs. It is possible to calculate the thrust of every arch and then analyse the cross ribs under a system of loads formed by the reactions of every elemental arch. The approach was first suggested by Frézier (1737) and it was applied many times since (for example by Dietlein 1823), in the context of La Hire's, incorrect, theory of arches). Again Heyman (1966, 1977) showed the systematic within the frame of Limit Analysis, Fig. 16.

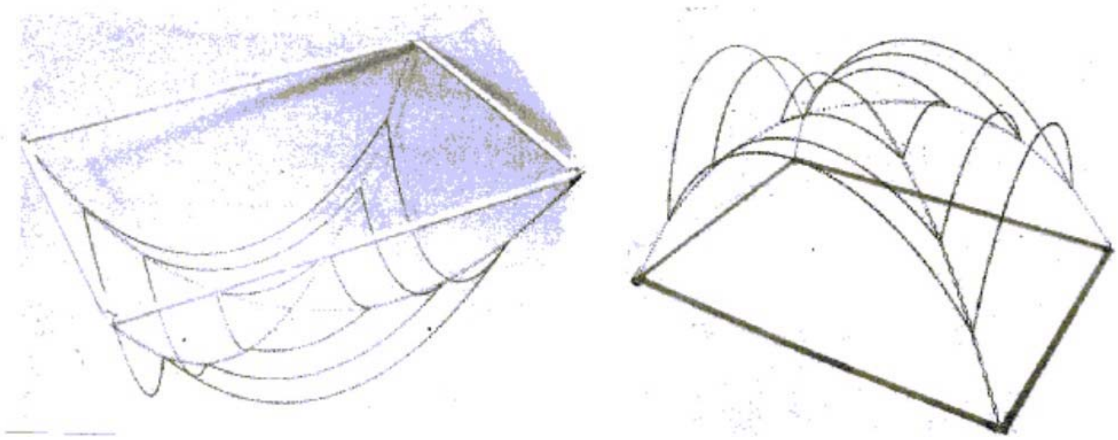

Figure 16 : Limit analysis of a gothic vault (Heyman 1995)

The first graphical analysis of cross vaults was made, apparently, by Wittmann (1879). Graphical statics allowed the engineers to make complex equilibrium analysis. Hanging models can also be used, either to understand physically the principle or as a tool for the architectural design, and Fig. 17 shows the essentials of a gothic vault equilibrium.

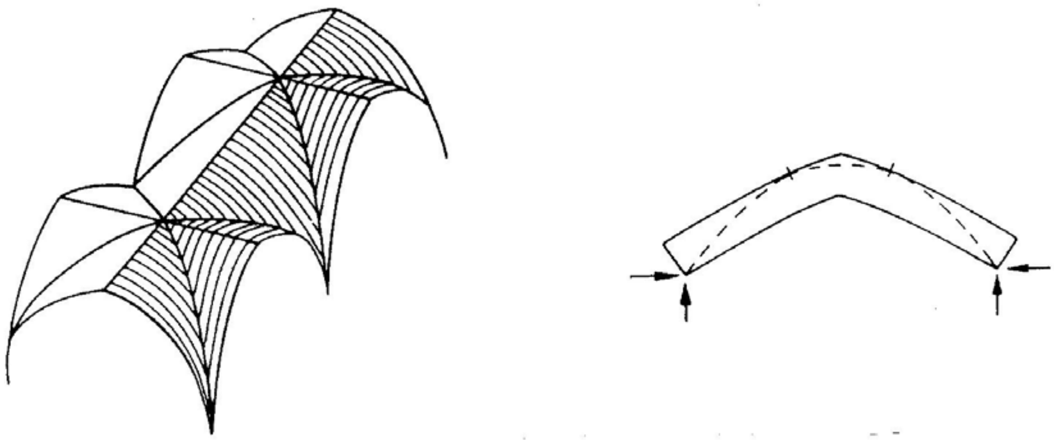

Figure 17 : Hanging model of a typical cross-vault (Beranek 1988)

Gaudi was the master of the use of models in the design of masonry structures. The crypt of the church of the Colonia Güell and the Sagrada Familia, both in Barcelona, are may be the best examples, Fig. 18. Gaudi used also graphical methods for the design of "equilibrated" masonry structures, Fig. 19.

As has been said the pattern of slicing depends on the form of the vault. In Figure 20 are represented various possibilities suggested by Mohrmann in his revision of Ungewitter´s *Lehrbuch der gotischen Konstruktionen* (Manual of Gothic Architecture) of 1890. The additions of Mohrmann constitute the most complete structural study of the gothic architecture.

Figure 18 : Gaudi=s funicular design for the church Colonia Güell (Rubió 1913)

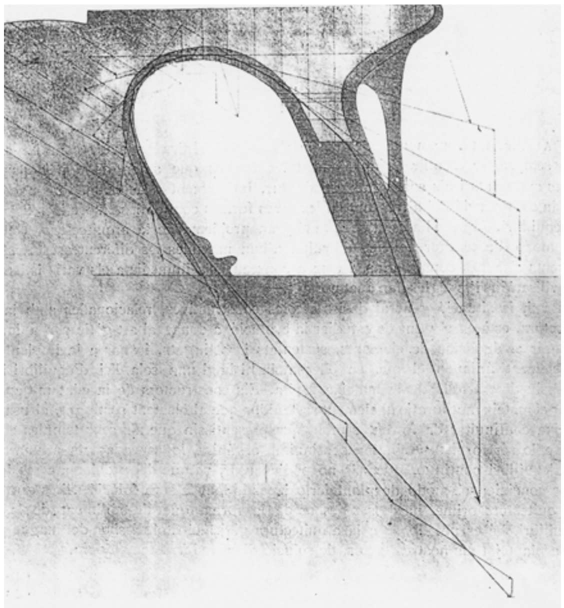

Figure 19 : Gaudi=s graphical design for the columns of the Park Güell (Rubió 1913)

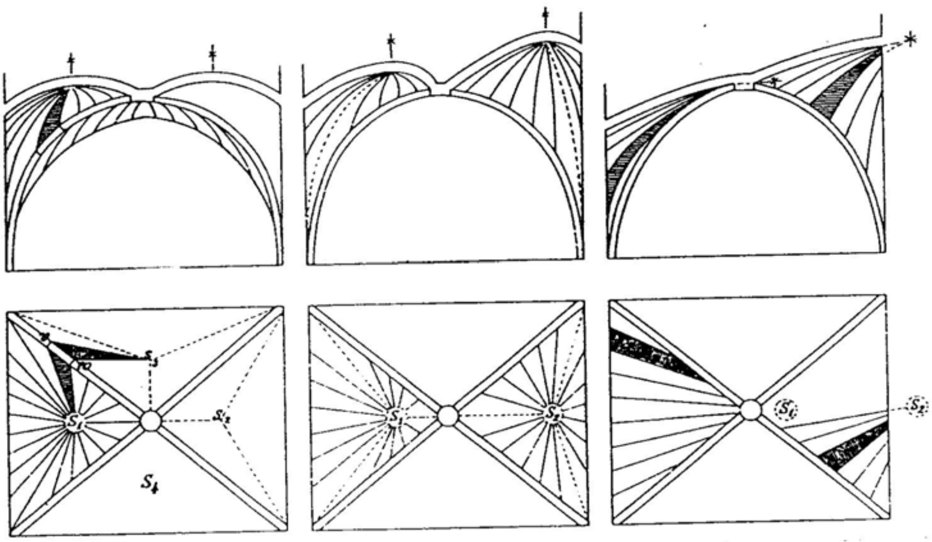

Figure 20 : Different patterns of slicing in the analysis of gothic vaults (Ungewitter and Mohrmann 1890)

Sometimes a certain pattern will lead to a surprisingly fast and simple way to estimate the vault thrust. Consider for example the problem of finding the thrust of a Spanish late-gothic vault. The overall geometry is in this case that of a domical vault, obtaining cutting a hemisphere with four vertical plans. The ribs are, then, roughly on a spherical surface, the only "creases" where a sharp discontinuity of curvature appears are at the transverse arches (which are thicker than the cross ribs in this type of vaults). Then, in Fig. 20 a different pattern of slicing is suggested. Vaults are cut with vertical planes parallel to the nave axis. The resulting elemental arches are all circular. If we consider an equivalent vault of the same weight and geometry, and of uniform thickness (which comprises the webs, keystones and ribs), we can deduce the vertical distribution of load over the transverse arches, only developing in vertical each of the semiarches (the horizontal component is annulled). The curve of loading is almost horizontal with a peak very near the buttress. It is evident that we can work with a uniform distribution and neglect this peak. Besides, the weight of the vault will be an order of magnitude greater that the weight of the transverse rib and we can check the stability of a "weightless" arch supporting a uniform load (obtained dividing the total weight of the bay by the span of the transverse arch). Of course, the line of thrust is parabolic and the thrust can be immediately calculated. The final check of the stability of the buttress can now be made. In Figure 21, can also be appreciated the relation between the material of the vault and that of the buttress system. The vault represents less than 10% of the masonry structure (this figure is typically between 5-10%, depending on the type of structure). The buttresses are important not only because they assure the global safety, most of the material consumption (and also the money) goes to them. Old master builders were right, again, considering buttress design the more important part of structural design (Huerta 1990).

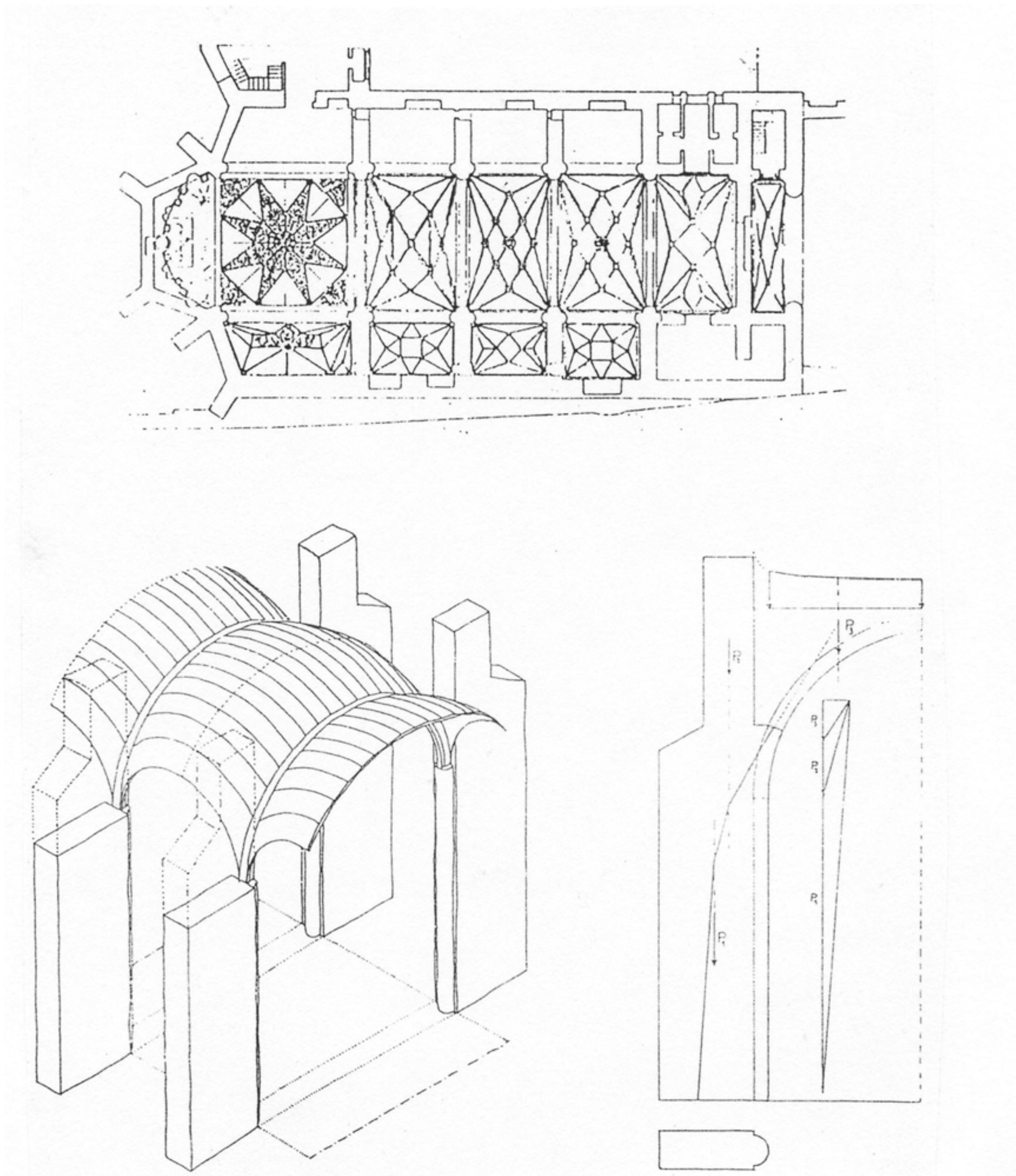

Figure 21 : Equilibrium analysis of the late-gothic vaults of the convent of Santo Domingo in Medina de Rioseco, Valladolid

It is obvious, then, that vault thrust depends fundamentally on the following factors: the overall dimensions of the bay, the thickness of the vault and the height of the vault. In fact with these data it will be possible to make a table. This global approach for obtaining directly the thrust of gothic vaults was first made by Michon (1857); detailed tables were compiled by Mohrmann (Ungewitter, 1890). Heyman (1995) includes a table based in Mohrmann's.

With reference to the cracks in gothic vaults, the same observations apply as in domes. Cracks should be interpreted as dividing the structures in a certain number of blocks which permit the movement imposed by the environment. It appears that the first study of gothic vault cracks was made by the French engineer Sabouret (1928, 1934). The first systematic study by Heyman (1983) (see also Barthel 1993). Fig. 22 shows Pol Abraham drawing of typical cracks and Heyman's interpretation.

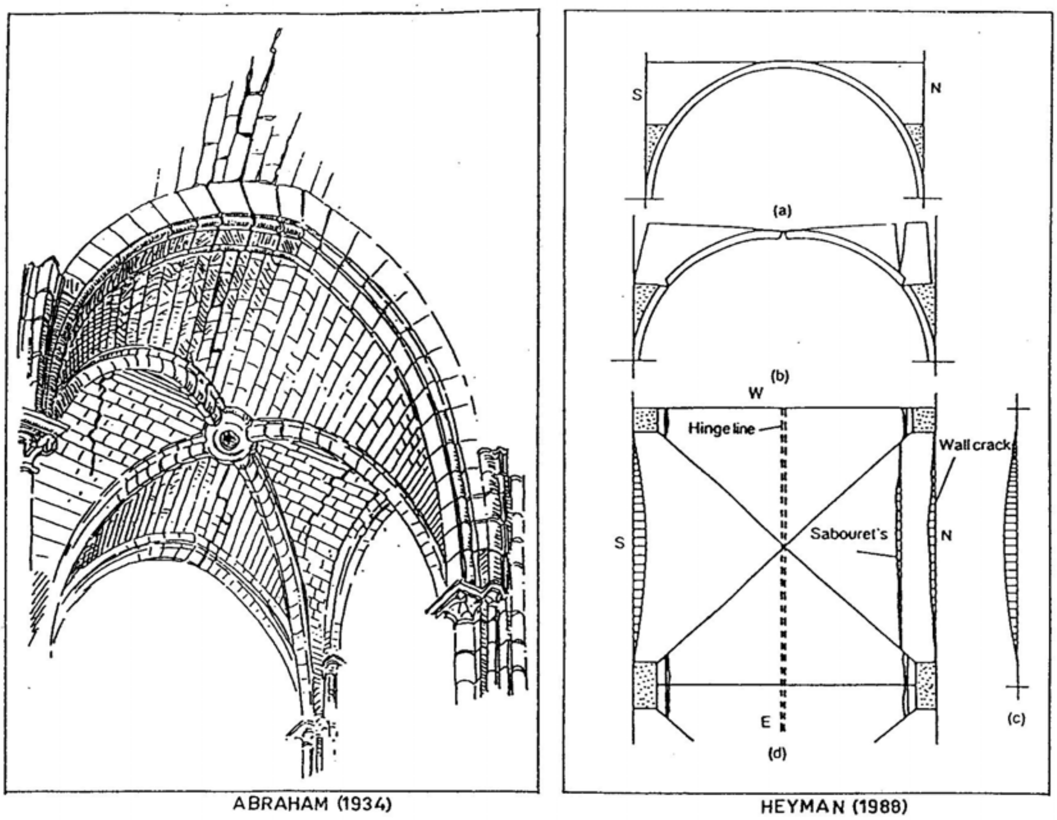

Figure 22 : Cracks in a gothic quadripartite vault

4.3 Gothic buttresses

It should be noted that in the past example the contribution of the walls between the buttresses has been ignored but the equilibrium state obtained is satisfactory, with the thrust at the base of the buttress well within the middle-third (geometrical safety factors are more restrictive for buttresses and the thrust should be contained within the middle-third; see for example, Rankine 1858). To ignore the wall's contribution is, of course, on the safe side. But, if after the analysis the equilibrium state is not satisfactory, we may want to make another hypothesis to account for the fact that the building still stands and have stood during perhaps four centuries. On the other side, to consider walls and buttresses forming a monolithic mass resisting the vault thrust may be too optimistic. The analyst, if the situation of the buttress system is considered critical, should study the problem with care, taking into account the internal constitution of the masonry (especially the bonding between walls and buttresses, or between counter-forts and walls).

The study of cracks and of the internal and external leaning of the system can help in this context. In the case of Fig. 10 (Huerta 1997) the leaning of the outer surface of the walls is greater than that of the inner surface. The conclusion is clear: there must exist an internal vertical crack dividing the wall in two. The resultant image is, of course, frightening. Also, the examination of the internal constitution of the masonry revealed unexpected results: at the top of the walls, where the vault thrust acts, the masonry is good, with an excellent lime mortar. At the base of the same walls the stones are "cemented" with clay. So the wall is not only anisotropic in the transverse direction but also in the vertical direction. In such a case, any analysis should consider with care the behaviour of the buttress system.

5. MASONRY BUILDINGS

In the study of historic buildings there are two main objectives: 1) to understand the way the structure behaves; 2) to understand the origin and significance of the cracks, if they are visible. Only with a good comprehension of both aspects can the engineer or architect emit a diagnostic and decide what to do (if there is anything to do at all). The first task implies the study of possible states of equilibrium. The second to imagine what kind of movements have given origin to the observed pattern of cracks.

The study of equilibrium can be best made first "identifying" the elements which compose the structure (in fact, the decision of what is structure and what is not is the very first part of analysis). In the gothic cathedrals the structure and its elements are very apparent; in Romanesque or Byzantine churches it is not so evident. It should be emphasised that this previous analysis has a paramount importance: an inadequate identification of the structure and its elements will be misleading. A common traditional division is between *vaults*, which thrust, and *buttresses* which resist the thrust (or counter-thrust). As we have seen, vaults and buttresses can be imagined in turn to be composed of different elements. Fortunately, the number of basic types of historic masonry structures is limited (since antiquity exists a "classification" which reveals itself in the different names used: domes, cross-vaults, groin vaults, cloister vaults, fan vaults, flying buttresses, . . .). Once the division in elements is made, the equilibrium states of each element are studied, which respect the essential condition for the material: the internal forces must be compressive. Finally, a global equilibrium is sought were all the elements interact in compression.

The second task is, in general, more difficult. The analyst should bear in mind the typical cracking patterns for the different types of arches and vaults and their relation to the abutment movements, and imagine by analogy, what kind of movements could have originated the actual pattern of cracks in the actual building. Complicated patterns produced by combined movements, will demand the analyst experience and insight

Great, complex, buildings were analysed only at the end of the XIXth century. The developing of graphical statics permitted, then, a good grasp of the equilibrium conditions and facilitated the analysis. As has been said, the most comprehensive and systematic study is in Ungewitter and Mohrmann (1890). Other contributions were concerned with the analysis of existing gothic cathedrals. For example, Benouville (1891) studied the cathedral of Beauvais, which has been studied also by Heyman (1967).

A very interesting contribution was the study of Rubió i Bellver (1912) on the structure of the cathedral of Palma de Mallorca. This cathedral is one of the greatest of gothic architecture. The central nave has a span of 19 m with 42 m of height. The main problem of design is the pillars of the main nave. They are very slender and they receive on top the thrust of the lateral nave (9 m of span). How would it be possible to obtain at the top of the pillar an almost vertical thrust? The answer is to charge the central nave with such a load that it is possible to obtain (with the help of the flying-buttresses) a force that will equilibrate the horizontal component of the thrust of the lateral nave, Fig. 23. The addition of weight increases, of course, the thrust to be transmitted by the main buttresses which are enormous. In Fig. 24 there is a vista of the exterior of the cathedral. The powerful image, like a big ship, is consequence of one decision of the gothic architect: to have slender pillars in a three-nave system with naves of different height.

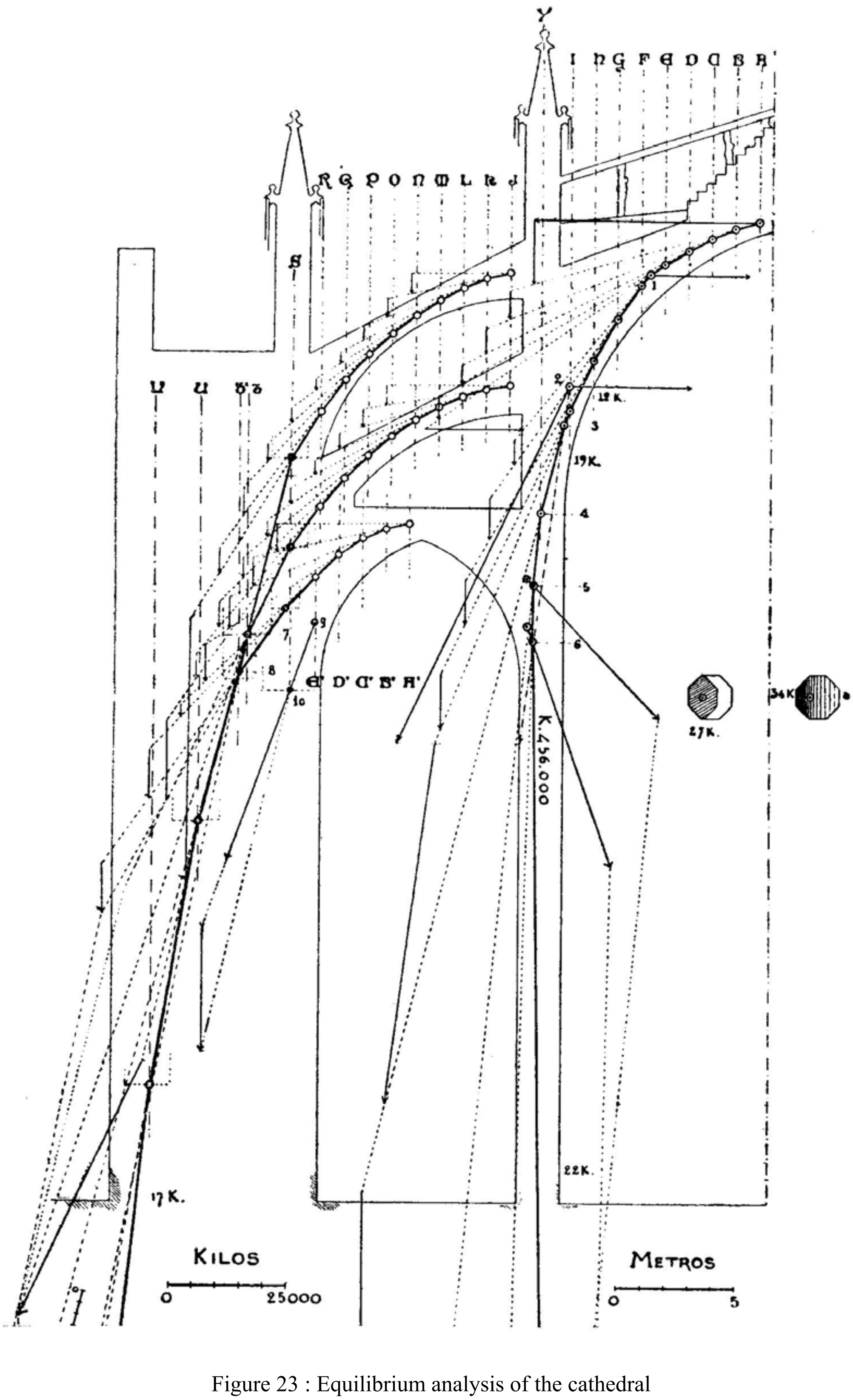

Figure 23 : Equilibrium analysis of the cathedral

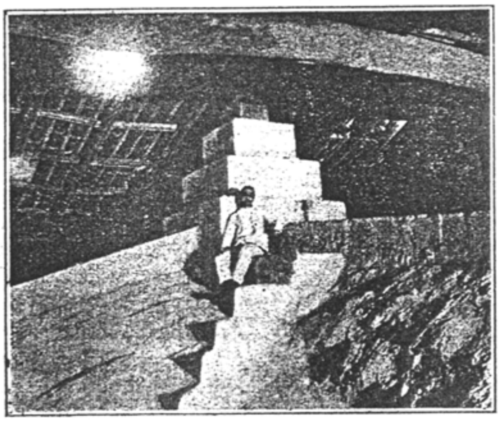

Figure 24 : Weights over the central keystone
of Palma de Mallorca (Rubiò, 1912) and transverse arches (Rubiò, 1912)

Figure 25 : Vista of the cathedral of Palma de Mallorca

Others studies can be mentioned. Zorn (1933) analysed the structure of the church of Sankt Martin in Landshut. Here the architect posed himself another "gothic" problem: how to maintain the slender vertical pillars in equilibrium when the lateral vault is much smaller that the central vault and the resultant thrust are unbalanced. The drawing of Zorn allows us to understand the artifice: above the transverse lateral arch there is a thick heavy wall of solid ashlar masonry and at its top charges part of the roof. The strategy is to augment the load of the lateral transverse arch so as to augment its thrust and, finally, be able to equilibrate, almost exactly, the thrust of the central nave. Indeed, in Zorn's drawing the load goes down almost vertically through the pillar.

As in Palma de Mallorca the gothic master is controlling the equilibrium playing with the loads as in a balance. In can be mentioned that usually neo-gothic churches are in a worse state as gothic churches. Bollig (1975) attributed this to the attitude of neo-gothic architects who copied gothic structures from the inside being ignorant of the necessity of superincumbent weights.

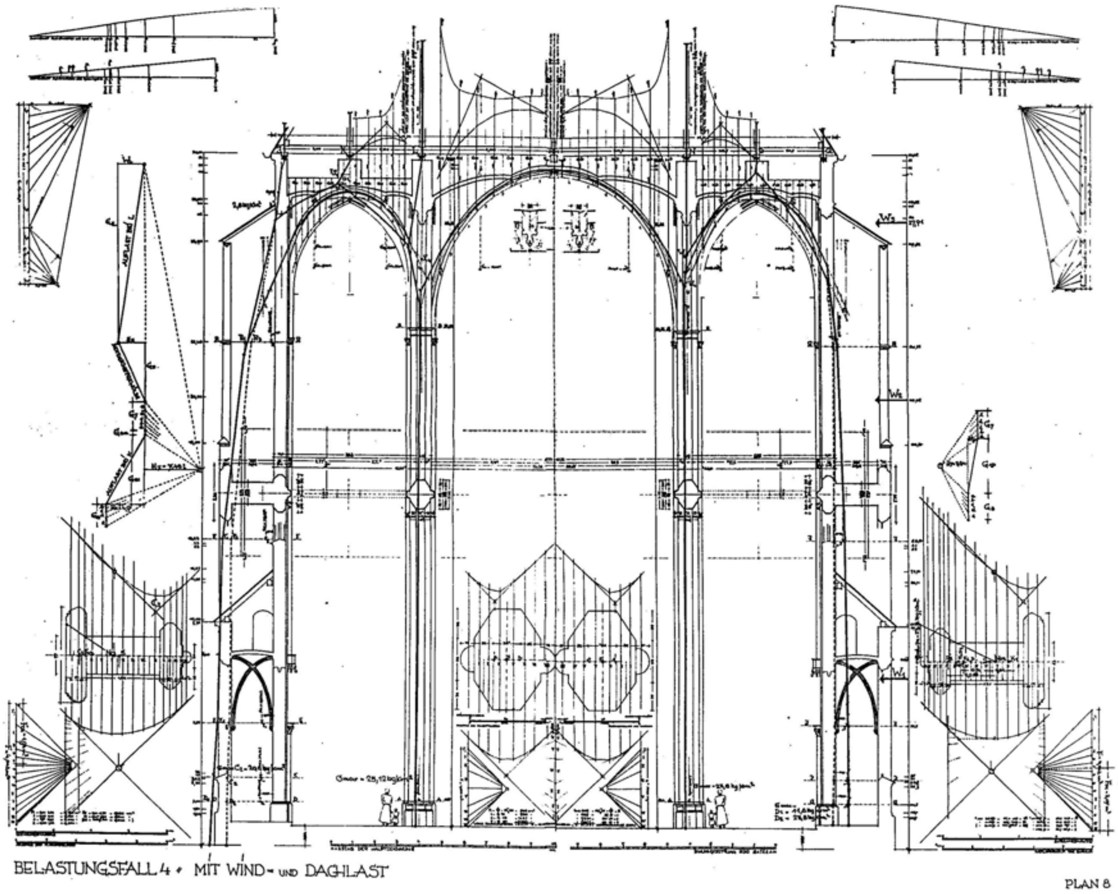

Figure 26 : Equilibrium analysis of the church of Sankt Martin in Landshut (Zorn 1933)

5. CONCLUSIONS

The equilibrium approach for the analysis and design of masonry structures has demonstrated to be the most adequate. It is embedded in the geometrical design rules of the old master builders. It was used by the great engineers of the XVIIIth and XIXth centuries. It follows directly from the Safe Theorem of Limit Analysis applied to masonry structures. This theorem constitutes, as professor Heyman (1999) has said, the "rock on which the whole theory of structural design is now seen to be based."

In considering only possible equilibrium solutions that respect the essential no-tension character of the material, the analyst is led to consider only the fundamental problems in question. No consideration is made of the changing and essentially unknowable boundary conditions. No affirmation is made about elastic properties of the masonry. The bare essentials of the complexity of the structure are now under consideration. The task is not easy, no computer program will give us a unique answer, but the problem present itself with all its fascinating complexity and richness. Now the analyst is in the situation to ask relevant questions and give meaningful answers.

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